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Abstract

We study transferable utility games with limited cooperation between the agents. The focus is on communication structures where the set of agents forms a circle, so that the possibilities of co-operation are represented by the connected sets of nodes of an undirected circular graph. Agents are able to cooperate in a coalition only if they can form a network in the graph. A single-valued solution which averages marginal contributions of each player is considered. We restrict the set of permutations, which induce marginal contributions to be averaged, to the ones in which every agent is connected to the agent that precedes this agent in the permutation. Staring at a given agent, there are two permutations which satisfy this restriction, one going clockwise and one going anticlockwise along the circle. For each such permutation a marginal vector is determined that gives every player his marginal contribution when joining the preceding agents. It turns out that the average of these marginal vectors coincides with the average tree solution. We also show that the same solution is obtained if we allow an agent to join if this agent is connected to some of the agents who is preceding him in the permutation, not necessarily being the last one. In this case the number of permutations and marginal vectors is much larger, because after the initial agent each time two agents can join instead of one, but the average of the corresponding marginal vectors is the same. We further give weak forms of convexity that are necessary and sufficient conditions for the core stability of all those marginal vectors and the solution. An axiomatization of the solution on the class of circular graph games is also given.

Keywords: Cooperative game, graph structure, average tree solution, Myerson value, core stability, convexity.

AMS subject classification: 90B18, 91A12, 91A43.

JEL code: C71.

1 Introduction

Cooperative game theory describes situations of cooperation between players. A cooperative game with transferable utility, TU-game for short, expresses such situations by a finite set of players and a characteristic function that assigns a worth to any subset of players, a coalition. Players within a coalition can freely divide the worth of the cooperation among themselves. The main focuses of TU-games are investigating under which conditions the players cooperate together to form the grand coalition of all players and how to divide the worth of this grand coalition into a payoff of each player.

TU-games assume that any coalition can be formed to cooperate and gain its worth of their cooperation, but in many economic situations there exist restrictions which prevent some coalitions from cooperating. A TU-game with this kind of situation is firstly introduced by Myerson [12] as a game with communication structure, called a graph game (cf. van den Brink, van der Laan and Pruzhansky [4], Baron, Béal, Rémila and Solal [1], etc.). It arises when the restriction is represented by an undirected graph in which the vertices represent the players and a link between two players shows that these players can communicate and are able to cooperate by themselves. More generally, one can assume that there is a collection of feasible coalitions, see Grabisch [7] for a survey. In this paper, we restrict our attention to the class of graph games.

A mapping is called a single-valued solution if it yields a payoff vector to each graph game. One of the most well-known single-valued solutions is the Myerson value [12], defined as the Shapley value [14] of the so-called Myerson restricted game. The Shapley value is the average of all marginal vectors of a game, where a marginal vector corresponds to a payoff vector for a permutation on the player set, in which each player gets as payoff the difference in worth of the set of players preceding him in the permutation with and without him. The Myerson value is characterized by efficiency and fairness, fair in the sense that if a link is deleted between two players, the Myerson value imposes the same loss on payoffs for each of these two players. Another characterization of the Myerson value is given by Myerson [13] and Borm, Owen and Tijs [3] on the class of cycle-free graph games. The position value, another single-valued solution of graph games, is introduced by Meessen [10] and Borm *et al.* [3]. The position value shares the Shapley value of the induced arc game, another graph restricted game which defines the worth to the power set of the set of links, among the players who own a link. It is characterized by Slikker [16] by efficiency and balanced link contributions. The latter means that for any pair of players, the total sum of the payoff losses of one player caused by breaking each link of the other player is the same for both players.

The average tree solution is introduced by Herings, van der Laan and Talman [8] on the class of cycle-free graph games. Unlike the Myerson value and the position value, this solution is not defined via some transformation of the original game but instead it is the average of the marginal vectors deduced from a specific collection of (rooted) spanning trees on the graph. For a cycle-free graph, every player induces exactly one spanning tree with himself as the root, and hence in case of n players the average tree solution is the average of n marginal vectors, while the

Myerson value is the average of $n!$ vectors and the position value uses $(n-1)!$ vectors on this class of graphs. On the class of cycle-free graph games Herings *et al.* [8] show that the average tree solution is characterized by efficiency and component fairness. The latter means that when a link between players is deleted the average loss of players in both resulting components is the same. Another characterization on this class of graph games is given by Mishra and Talman [11]. They show that the solution is completely characterized by efficiency, the dummy property, linearity, strong symmetry, and independence in unanimity games. The last property is not satisfied by the Myerson value and says that if the minimum winning network of a unanimity game gets bigger by a player, then the payoff of any other player in the network not being linked to this player does not change.

Herings, van der Laan, Talman and Yang [9] generalize the average tree solution to the class of arbitrary graph games. Given a graph, they define a collection of admissible spanning trees as the ones where each player has in each component of his subordinates one successor. This selects trees on the graph which describe how the players can be partially ordered in such a way that if there is a communication link between two players, one of them should be a subordinate of the other. When the underlying graph has cycles, and therefore more communication links, there are typically more ways for players to communicate and the number of admissible spanning trees becomes larger. Baron *et al.* [1] give an axiomatization on the class of arbitrary graph games as a unique solution satisfying efficiency and $\mathcal{T}$-hierarchy. The latter property means that in a unanimity graph game for a network the payoff is only explained by how often a player is a root in the smallest subtree that contains the network under all admissible spanning trees.

We study solutions on the class of circle graph games where the underlying graph is assumed to be a circle. Players could be firms or cities situated along a lakeshore or circular pipeline where players can only be connected to their two direct neighbors, one located on each side. A subset of players is in such a setting only able to cooperate if it consists of consecutive nodes on the circle. As described above the Shapley value for a TU game, i.e. with full communication, is the average of the marginal vectors corresponding to, in case of n players, all $n!$ permutations on the player set. For a circle graph game we propose to take as solution the average of the marginal vectors which corresponds to permutations in which each player has a communication link with the player preceding him in the permutation. The idea is that if a player is not connected to the player that is immediately preceding him in the permutation then this player is not able to cooperate with his preceding players and therefore doesn't receive his marginal contribution. It turns out that on the class of circle graph games the average of the marginal vectors of these admissible permutations is precisely the average tree solution introduced in [9]. The average tree solution therefore allocates to each player the average of his marginal contributions when he joins clockwise or anticlockwise any of the networks he is connected to. If there are n players there are $2n$ of such admissible permutations, each yielding a different marginal vector. Instead of looking only at permutations in which every player is linked to his immediate predecessor in the permutation, one could also argue that a player may join the predecessors in the permutation if he is connected to at least one them, not being necessarily the last one. The idea here is that if a player is linked to some of the players that precede him in the permutation, he is able to cooperate with them and get his marginal contribution. Since the starting agent can be any agent and every time one of two agents can join until the last agent is left, the number of permutations is equal to $2^{n-2}n$ in case of n players. Each such permutation leads to a different marginal vector and one may take the average of these marginal vectors as solution concept. It appears that this solution is equal to the Shapley value introduced by Bilbao and Ordóñez [2] and for the class of circle graph games it coincides with the solution proposed before. Although the two sets of permutations and of marginal vectors differ for both solutions, the resulting payoff distribution is precisely the same.

We further give an expression of the solution on the class of unanimity circle graph games in terms of the representation power of each player, introduced in [8] for the class of cycle-free graph games. We also propose an axiomatic characterization of the solution on the class of circle graph games, similar to the one in [11] for the class of cycle-free graph games. We show that the solution is completely characterized by efficiency, the dummy property, linearity, symmetry at end players in unanimity games, and independence in unanimity games. Symmetry at end players in unanimity games says that in unanimity circle graph games the players that are the end points of the minimum winning coalition receive the same payoff. This property is also satisfied by the Myerson value.

The stability of the solution on the class of circle graph games is studied as well. On the class of cycle-free graph games, Herings *et al.* [8] show that superadditivity is a sufficient condition under which the average tree solution is an element of the core, where a game is superadditive if the worth of the union of any two disjoint coalitions is at least equal to the total worth of both. This condition is further weakened by Talman and Yamamoto [17] with the notion of satellites. On the class of all graph games, Herings *et al.* [9] introduce the notion of link convexity, which is weaker than superadditivity on the cycle-free graph game but stronger than the condition found in [17]. In this paper we introduce the notion of circular-convexity on the class of circle graph games. Circular-convexity is weaker than convexity but stronger than superadditivity. It is well known that for TU games convexity is equivalent to the property that all marginal vectors lie in the core and therefore also the Shapley value. We show that for a circle graph game, circular-convexity is a necessary and sufficient condition to guarantee that every admissible marginal vector for the average tree solution is an element of the core and therefore also the average tree solution. A stronger version of circular-convexity is also given as a necessary and sufficient condition for every marginal vector for the Shapley value introduced in [2] to be in the core. We further give a necessary and sufficient condition for the solution itself to be in the core, called average-circular-convexity. This condition is not necessarily stronger than superadditivity, and weaker than circular-convexity. We also illustrate that the Myerson value may not be in the core if the game is circular-convex.

This paper is organized as follows. Section 2 is a preliminary section on circle graph games. In Section 3 the solution concepts for the class of circle graph games are introduced. In Section 4 an axiomatic characterization is given. In Section 5 the notions of circular-convexity are introduced and their relationships to the stability of the solution concepts are explained. Section 6 concludes.

2 Circle graph games

Consider a finite number of nodes or agents located on a circle. The nodes could for example be villages along a lake shore or around a mountain, shopping malls along a ring road of a city, or companies connected to a circular pipeline. Let the set $N = \{1, ..., n\}$ denote the set of nodes, with $n \geq 3$. Given the location of the nodes on a circle, we assume without loss of generality that each node $i \in N$ has two neighbors, $i - 1$ and $i + 1$, and that there is a link between any node and each of his neighbors, where $i - 1 = n$ when $i = 1$ and $i + 1 = 1$ when $i = n$. A link between two agents has as interpretation that the two agents are able to cooperate in order to get their joint worth. Let L denote the set of links between any two neighbors, that is $L = \{\{i, i + 1\}|\ i = 1, \ldots, n\}$. The pair (N, L) is an undirected graph with the set of agents N as the set of nodes and the set of links L as the set of edges. A subset, or coalition, $S \in 2^N$ is connected, or a network, in (N, L) if for any $i \in S$ and $j \in S$, $j \neq i$, there is a sequence of nodes $(i_1, i_2, \ldots, i_k)$ in S such that $i_1 = i$, $i_k = j$ and $\{i_h, i_{h+1}\} \in L$ for $h = 1, \ldots, k - 1$. The collection of networks in (N, L) is denoted $C^L(N)$. By definition, the empty set $\emptyset$, every singleton $\{i\}$, $i \in N$, and the grand coalition N are networks in (N, L). For $S \in 2^N$, the subset of links $L(S) \subset L$ is defined as $L(S) = \{\{i, j\} \in L|\ i, j \in S\}$, being the subset of L of links that can be established within S. The graph $(S, L(S))$ is a subgraph of (N, L). A component of $(S, L(S))$ is a network in $(S, L(S))$ that is maximally connected in $(S, L(S))$. The collection of components of the subgraph $(S, L(S))$ of (N, L) is denoted $\hat{C}^L(S)$. A characteristic function $v : 2^N \to \mathbb{R}$, with $v(\emptyset) = 0$, defines the worth of a coalition $S \in 2^N$. Only the worth $v(S)$ of a network $S \in C^L(N)$ can be freely transferred among the members of S. The triple (N, v, L) is called a circle graph game. Following Myerson [12], the restricted characteristic function $v^L : 2^N \to \mathbb{R}$ of v is defined as

$$v^L(S) = \sum_{K \in \hat{C}^L(S)} v(K),\ S \in 2^N.$$

The pair (N, v^L) is a transferable utility game (TU-game) and is called the restricted game of (N, v, L). The class of all TU-games (N, v) with characteristic function $v : 2^N \to \mathbb{R}$ is denoted by $\mathcal{G}$ and the class of all circle graph games (N, v, L) is denoted by $\mathcal{G}^c$.

Given a circle graph game $(N, v, L) \in \mathcal{G}^c$, the problem is how to allocate the worth $v(N)$ of the grand coalition to the agents. An allocation $x = (x_1, ..., x_n) \in \mathbb{R}^n$ is a payoff vector and assigns payoff x_i to agent $i \in N$. For a payoff vector x, we often write $x(S)$ or x_S instead of $\sum_{i \in S} x_i$ for $S \in 2^N$. A solution on the class of circle graph games is a mapping that assigns to every circle graph game a (possibly empty) set of payoff vectors. The core of a circle graph game $(N, v, L) \in \mathcal{G}^c$ is defined as

$$C(N, v, L) = \{x \in \mathbb{R}^n \mid\ x(N) = v(N),\ x(S) \geq v(S),\ S \in C^L(N)\}.$$

The core is the set of allocations that are efficient, $x(N) = v(N)$, and are not opposed by any network, that is $x(S) \geq v(S)$ for all $S \in C^L(N)$. For the class of TU-games, in which all coalitions are networks, the core is introduced by Gillies [6] and is for a game $(N, v) \in \mathcal{G}$ given by

$$C(N, v) = \{x \in \mathbb{R}^n \mid \ x(N) = v(N),\ x(S) \geq v(S),\ S \in 2^N\}.$$

Notice that for any circle graph game $(N, v, L) \in \mathcal{G}^c$ it holds that $C(N, v, L) = C(N, v^L)$.

A single-valued solution on the class of circle graph games is the Myerson value. For a circle graph game $(N, v, L) \in \mathcal{G}^c$, it is the Shapley value of its restricted game (N, v^L), see [12], and is defined as follows. Let $\Pi(N)$ be the set of permutations on N. For a permutation $\sigma = (\sigma(1), \ldots, \sigma(n))$ in $\Pi(N)$ the marginal vector $m^\sigma(N, v, L)$ assigns payoff

$$m^\sigma_{\sigma(k)}(N, v, L) = v^L(\{\sigma(1), \ldots, \sigma(k)\}) - v^L(\{\sigma(1), \ldots, \sigma(k-1)\}) \tag{2.1}$$

to agent $\sigma(k)$, $k = 1, \ldots, n$. The Myerson value, $\mu(N, v, L)$, is then the average of all $n!$ marginal vectors, i.e.,

$$\mu(N, v, L) = \frac{1}{n!} \sum_{\sigma \in \Pi(N)} m^\sigma(N, v, L).$$

3 Solutions for circle graph games

The Shapley value of a TU-game in which any coalition is a network can be interpreted as follows. To form the grand coalition, agents enter a room randomly one-by-one and if an agent enters he connects to the last person who entered before and he receives his marginal contribution for joining the agents who are already present in the room. In this way a permutation $\sigma = (\sigma(1), \ldots, \sigma(n))$ is obtained in which first agent $\sigma(1)$ enters, which can be any of the n agents, and this agent receives his worth $v(\{\sigma(1)\})$, the minimum amount to let him stay in the room. Then agent $\sigma(2)$ enters, which can be any of the remaining $n-1$ agents, he receives as payoff his marginal contribution $v(\{\sigma(1), \sigma(2)\}) - v(\{\sigma(1)\})$ when joining agent $\sigma(1)$, otherwise the two agents would not stay together in the room, and agent $\sigma(2)$ connects to agent $\sigma(1)$ to form the ordering $(\sigma(1), \sigma(2))$. Then from the remaining $n-2$ agents agent $\sigma(3)$ enters, gets as payoff his marginal contribution $v(\{\sigma(1), \sigma(2), \sigma(3)\}) - v(\{\sigma(1), \sigma(2)\})$, otherwise the three agents would not stay together in the room, and connects to agent $\sigma(2)$, the last agent who joined before, to form the ordering $(\sigma(1), \sigma(2), \sigma(3))$, and so on, until the last agent, $\sigma(n)$, enters, gets his marginal contribution $v(N) - v(N \setminus \{\sigma(n)\})$, and connects to agent $\sigma(n-1)$, the last agent who joined before, to complete the ordering σ in forming the grand coalition N. In general, for $k = 2, \ldots, n$, when $k-1$ agents, $\sigma(1), \ldots, \sigma(k-1)$, have entered the room before, agent $\sigma(k)$, one of the remaining $n-k+1$ agents, enters. This agent gets as payoff $v(\{\sigma(1), \ldots, \sigma(k)\}) - v(\{\sigma(1), \ldots, \sigma(k-1)\})$, being his contribution when joining the agents in the room, otherwise they would leave the room, and connects to agent $\sigma(k-1)$, the last agent who entered before, to form the ordering $(\sigma(1), \ldots, \sigma(k))$. The Shapley value is the average of all such marginal contributions.

In a circle graph it is not the case that every agent is connected to every other agent. We assume that if an agent enters the room as described above and he is not connected to the last agent that entered before, then the grand coalition cannot be formed. The idea is that if there is no link between the agent who enters and the last agent who entered before, the entering agent is not able to communicate and therefore cannot form a coalition with the agents who entered before. In this way only orderings σ are able to form the grand coalition in which, for $k = 2, \ldots, n$, once the $k-1$ agents $\sigma(1), \ldots, \sigma(k-1)$ have entered the room in this order, agent $\sigma(k)$ will only enter and stay in the room if he is connected to the last agent that entered before, being agent $\sigma(k-1)$. In this case he receives as payoff his marginal contribution $v(\{\sigma(1), \ldots, \sigma(k)\}) - v(\{\sigma(1), \ldots, \sigma(k-1)\})$, otherwise the k agents would not stay together in the room, and agent $\sigma(k)$ connects to agent $\sigma(k-1)$ to form the ordering $(\sigma(1), \ldots, \sigma(k))$. In other words, we assume that only if for all $k = 2, \ldots, n$ node $\sigma(k)$ is linked to node $\sigma(k-1)$, then the grand coalition N can be formed through the ordering σ and every agent receives his marginal contribution. If, for at least one $k \in N$, agent $\sigma(k)$ is not connected to $\sigma(k-1)$, then we assume that the grand coalition cannot be formed through the ordering σ. In case agent $i = \sigma(1)$ enters the room first, there are just two agents, agent $i-1$ (agent n when $i = 1$) and agent $i+1$ (agent 1 when $i = n$), being connected to agent i and who therefore may enter the room to join agent i. After one of these two agents enters, there is only one of the remaining $n-2$ agents who can enter and join the agent who entered before, and so on, until the last remaining agent enters. This leads to $2n$ different orderings, or permutations, through which the grand coalition can be formed. Let us call these permutations admissible. For each node $i \in N$ there are two admissible permutations σ with $\sigma(1) = i$, denoted $\sigma_1^i = (i, i+1, \ldots, n, 1, \ldots, i-1)$ and $\sigma_2^i = (i, i-1, \ldots, 1, n, \ldots, i+1)$. The set of admissible permutations is then given by

$$\Pi^a(N) = \{\sigma_1^i \mid i = 1, \ldots, n\} \cup \{\sigma_2^i \mid i = 1, \ldots, n\}.$$

Given a circle graph game $(N, v, L) \in \mathcal{G}^c$, to any admissible permutation $\sigma \in \Pi^a(N)$ a marginal vector $m^\sigma(N, v, L)$ corresponds and assigns payoff

$$m^\sigma_{\sigma(k)}(N, v, L) = v(\{\sigma(1), \ldots, \sigma(k)\}) - v(\{\sigma(1), \ldots, \sigma(k-1)\})$$

to agent $\sigma(k)$, $k = 1, \ldots, n$. Notice the difference from equation (2.1), where the worths of the restricted game v^L are taken because coalitions may not be networks. As solution concept we take the average of these $2n$ marginal vectors,

$$\frac{1}{2n} \sum_{\sigma \in \Pi^a(N)} m^\sigma(N, v, L).$$

We will show now that this solution coincides with the average tree solution. The average tree solution has been introduced on the class of arbitrary graph games by Herings *et al.* [9] and is defined on the class of circle graph games as follows.

An n-tuple $B = (B_1, \ldots, B_n)$ of networks in a circle graph (N, L) is admissible if there is some $r \in N$ such that $B_r = N$ and for all $i \in N$ it holds that $i \in B_i$ and, for some $j \in N$,

$\{i,j\} \in L$ and $B_j = B_i \setminus \{i\}$. Let $\mathcal{B}^L$ denote the collection of admissible n-tuple of networks in (N,L). Given a circle graph game $(N,v,L) \in \mathcal{G}^c$, to an admissible $B \in \mathcal{B}^L$ the marginal vector $m^B(N,v,L)$ corresponds, defined by

$$m_i^B(N,v,L) = v(B_i) - v(B_i \setminus \{i\}),\ i \in N.$$

The average tree solution, $AT(N,v,L)$ is then defined as the average of the marginal vectors corresponding to all admissible n-tuples of networks in (N,L), i.e.,

$$AT(N,v,L) = \frac{1}{|\mathcal{B}^L|} \sum_{B \in \mathcal{B}^L} m^B(N,v,L).$$

Theorem 3.1 *For any circle graph game $(N,v,L) \in \mathcal{G}^c$ it holds that*

$$AT(N,v,L) = \frac{1}{2n} \sum_{\sigma \in \Pi^a(N)} m^\sigma(N,v,L).$$

Proof. Take any $\sigma \in \Pi^a(N)$ and suppose $\sigma = \sigma_1^i$ for some $i \in N$. Define $B_k = \{i,\ldots,k\}$ for $k = i,\ldots,n$, and $B_k = \{i,\ldots,n,1,\ldots,k\}$ for $k = 1,\ldots,i-1$. Then $B = (B_1,\ldots,B_n)$ is an admissible n-tuple of networks in (N,L), satisfying $m^B(N,v,L) = m^{\sigma_1^i}(N,v,L)$. Similarly, when $\sigma = \sigma_2^i$ for some $i \in N$, define $B_k = \{k,\ldots,i\}$ for $k = 1,\ldots,i$, and $B_k = \{k,\ldots,n,1,\ldots,i\}$ for $k = i+1,\ldots,n$. Then again this $B = (B_1,\ldots,B_n)$ is an admissible n-tuple of networks in (N,L), satisfying $m^B(N,v,L) = m^{\sigma_2^i}(N,v,L)$. Therefore every permutation in $\Pi^a(N)$ corresponds to a unique admissible n-tuple of networks in (N,L). Next, let $B = (B_1,\ldots,B_n)$ be an admissible n-tuple of networks in (N,L). Then there exists unique $i \in N$ such that $B_i = N$. Consider the set $B_i \setminus \{i\} = N \setminus \{i\}$. This set has two elements that are linked to i, namely $i-1$ and $i+1$, where $i-1 = n$ when $i = 1$ and $i+1 = 1$ when $i = n$. So, $B_i \setminus \{i\}$ is either B_{i+1} (B_1 when $i = n$) or B_{i-1} (B_n when $i = 1$). Suppose $B_i \setminus \{i\} = B_{i+1}$. Then, when $i < n$, $i+2$ is the only element of $B_{i+1} \setminus \{i+1\}$ that is linked to $i+1$ and so $B_{i+1} \setminus \{i+1\} = B_{i+2}$, and, when $i = n$, 2 is the only element of $B_1 \setminus \{1\}$ that is linked to 1, and so on. In every further step there is only one element in $B_k \setminus \{k\}$ that is linked to k, and that is the element $k+1$ and so $B_k \setminus \{k\} = B_{k+1}$, for $k = i+1,\ldots,n,1,\ldots,i$. From this it follows that $m^B(N,v,L) = m^{\sigma_2^{i-1}}(N,v,L)$. Similarly, if $B_i \setminus \{i\} = B_{i-1}$, it holds that $m^B(N,v,L) = m^{\sigma_1^{i+1}}(N,v,L)$. Therefore every admissible n-tuple of networks in (N,L) corresponds to a unique permutation in $\Pi^a(N)$, which completes the proof. $\square$

The theorem says that on the class of circle graph games the average tree solution is equal to the average of all marginal vectors that correspond to permutations on the players set in which every two consecutive players of the permutation are neighbors of each other. Only such permutations are assumed to be able to form the grand coalition, coming from the interpretation of the Shapley value described above.

The Shapley value of a TU-game can also be interpreted in a slightly different way. To form the grand coalition, agents enter a room randomly one-by-one and if an agent enters he just joins the set of agents that are already present in the room and he receives his marginal contribution. In this interpretation an entering agent does not connect to the last agent who entered before but he just joins the set of agents who entered before. The Shapley value can be seen as the average of such marginal contributions.

Under the circular communication structure with this interpretation, it holds for ordering σ that when the $k-1$ agents $\sigma(1), \ldots, \sigma(k-1)$ have entered the room, the next agent, agent $\sigma(k)$, can only enter and gets his marginal contribution if he is connected to at least one of the $k-1$ preceding agents, not necessarily being agent $\sigma(k-1)$. The idea is that a player can only join a coalition to form a larger coalition if he is able to communicate with at least one member of that coalition. Let us call such orderings compatible. Observe that any admissible ordering is also compatible. After the first agent $\sigma(1)$, which can be any of the n agents, enters the room, the second agent who enters, agent $\sigma(2)$, can be any of the two neighbors of $\sigma(1)$, which is the same in the previous interpretation. However, agent $\sigma(3)$, who enters next, can be either the remaining neighbor of agent $\sigma(1)$, not being agent $\sigma(2)$, or the remaining neighbor of agent $\sigma(2)$, not being agent $\sigma(1)$. In general, if $\sigma(1), \ldots, \sigma(k-1)$ have entered, then agent $\sigma(k)$, who is entering the room next, is connected to one of the two end points of the induced network on the set $\{\sigma(1), \ldots, \sigma(k-1)\}$.

Given the first agent $\sigma(1) \in N$, there are two choices of $\sigma(2)$ for being compatible with $\sigma(1)$. In general, for $2 \leq k \leq n-1$, there are two choices of $\sigma(k)$ for being compatible with $(\sigma(1), \sigma(2), \ldots, \sigma(k-1))$. For the case $k = n$, the last agent, $\sigma(n)$, is uniquely determined, while the first agent, $\sigma(1)$, can be chosen arbitrarily among n agents. This leads to $2^{n-2}n$ different compatible orderings, or permutations, through which the grand coalition can be formed. In order to describe the collection of compatible permutations, first define the set of predecessors of any agent $i \in N$ in permutation $\sigma \in \Pi(N)$ as

$$P_\sigma(i) = \{h \in N| \ \sigma^{-1}(h) < \sigma^{-1}(i)\}.$$

The set of compatible permutations can be defined as

$$\Pi^c(N) = \{\sigma \in \Pi(N)| P_\sigma(i) \in C^L(N) \ \ \forall \ i \in N\}.$$

Given a circle graph game $(N, v, L) \in \mathcal{G}^c$, to any compatible permutation $\sigma \in \Pi^c(N)$ a marginal vector $m^\sigma(N, v, L)$ corresponds and assigns payoff

$$m_i^\sigma(N, v, L) = v(P_\sigma(i) \cup \{i\}) - v(P_\sigma(i)) \tag{3.1}$$

to agent $i \in N$. As solution concept we may take the average of the marginal vectors induced by all compatible orderings,

$$\frac{1}{2^{n-2}n} \sum_{\sigma \in \Pi^c(N)} m^\sigma(N, v, L).$$

We will show now that this solution coincides with the Shapley value introduced by Bilbao and Ordóñez [2] on the class of games with augmenting systems, which contains the class of (circle) graph games. An augmenting system on the set N is defined as a pair $(N, \mathcal{F})$ where

$\mathcal{F} \subseteq 2^N$ satisfies: $\emptyset \in \mathcal{F}$; for $S, T \in \mathcal{F}$ with $S \cap T \neq \emptyset$, then $S \cup T \in \mathcal{F}$; for $S, T \in \mathcal{F}$ with $S \subset T$, there exists $i \in T \setminus S$ such that $S \cup \{i\} \in \mathcal{F}$. Observe that $(N, C^L(N))$ is an augmenting system that represents the collection of networks of the circle graph (N, L). A game on the augmenting system $(N, \mathcal{F})$ is a triple $(N, v, \mathcal{F})$ where $\mathcal{F}$ describes the set of feasible coalitions. An ordering $\rho = (\rho(1), \ldots, \rho(n)) \in \Pi(N)$ is compatible on $(N, \mathcal{F})$ with $N \in \mathcal{F}$ if $\{\rho(1), \ldots, \rho(k)\} \in \mathcal{F}$ for all $k = 1, \ldots, n$, and corresponds one-to-one to a maximal chain R in $\mathcal{F}$, being a collection of coalitions in $\mathcal{F}$ ordered with respect to set inclusion that is not contained in any larger chain in $\mathcal{F}$. Let $Ch(\mathcal{F})$ denote the set of maximal chains in $\mathcal{F}$. Each maximal chain $R \in Ch(\mathcal{F})$ corresponding to compatible ordering ρ on $(N, \mathcal{F})$ induces a marginal vector which assigns

$$m^R_{\rho(i)}(N, v, \mathcal{F}) = v(\{\rho(1), \ldots, \rho(i)\}) - v(\{\rho(1), \ldots, \rho(i-1)\})$$

to agent $\rho(i) \in N$. Then the Shapley value of a game $(N, v, \mathcal{F})$ is defined in [2] as the average of the marginal vectors induced from all maximal chains in $\mathcal{F}$, i.e.,

$$Sh(N, v, \mathcal{F}) = \frac{1}{|Ch(\mathcal{F})|} \sum_{R \in Ch(\mathcal{F})} m^R(N, v, \mathcal{F}).$$

Theorem 3.2 *For any circle graph game $(N, v, L) \in \mathcal{G}^c$ it holds that*

$$Sh(N, v, C^L(N)) = \frac{1}{2^{n-2}n} \sum_{\sigma \in \Pi^c(N)} m^\sigma(N, v, L).$$

Proof. The circle graph (N, L) is equivalent to an augmenting system $(N, \mathcal{F})$ with set of feasible coalitions equals to $\mathcal{F} = C^L(N)$. For any maximal chain $R \in Ch(C^L(N))$ with corresponding compatible ordering ρ on $(N, C^L(N))$, ρ is an element in $\Pi^c(N)$. Conversely, any compatible permutation $\sigma \in \Pi^c(N)$ is a compatible ordering on $(N, C^L(N))$ corresponding to some maximal chain R in $C^L(N)$. Therefore, for any maximal chain R in $Ch(C^L(N))$ with corresponding compatible permutation $\sigma \in \Pi^c(N)$ it holds for all $i \in N$ that

$$\begin{aligned}
m^\sigma_i(N, v, L) &= v(P_\sigma(i) \cup \{i\}) - v(P_\sigma(i)) \\
&= v(\{h \in N | \sigma^{-1}(h) < \sigma^{-1}(i)\} \cup \{i\}) - v(\{h \in N | \sigma^{-1}(h) < \sigma^{-1}(i)\}) \\
&= v(\{\sigma(1), \ldots, \sigma(\sigma^{-1}(i))\}) - v(\{\sigma(1), \ldots, \sigma(\sigma^{-1}(i) - 1)\}) \\
&= m^R_i(N, v, C^L(N)).
\end{aligned}$$

□

Thus far, we have obtained two solution concepts on the class of circle graph games, following from two different interpretations of the Shapley value on the class of TU-games. The first one, which turns out to be the average tree solution introduced in [9], is the average of $2n$ marginal vectors, while the other solution, which turns out to be the Shapley value as introduced in [2], is the average of $2^{n-2}n$ marginal vectors. Generically, all latter marginal vectors are different and they contain the former ones. Nevertheless the two averages are the same, that is the two solution concepts introduced above coincide on the class of circle graph games.

For $i, j \in N$, let S_i^j denote the coalition containing all players from i to j, i.e., $S_i^j = \{i, i+1, \ldots, j\}$ if $j \geq i$ and $S_i^j = \{i, i+1, \ldots, n, 1, \ldots, j\}$ if $i > j$. Notice that $S_i^{i-1} = N$, where $i - 1 = n$ when $i = 1$, and $S_i^i = \{i\}$, $i = 1, \ldots, n$. For any $i, j \in N$ we say that S_i^j contains clockwise the elements between i and j and anticlockwise the elements between j and i.

Lemma 3.3 *For any circle graph game $(N, v, L) \in \mathcal{G}^c$ the average tree solution $AT(N, v, L)$ is equal to*

$$AT_j(N, v, L) = \frac{1}{2n} \sum_{i \in N} \Big(v(S_i^j) - v(S_i^j \setminus \{j\}) + v(S_j^i) - v(S_j^i \setminus \{j\}) \Big), \;\; j \in N.$$

Proof. For $j \in N$ it holds that

$$\begin{aligned} AT_j(N, v, L) &= \frac{1}{2n} \sum_{i \in N} \big(m_j^{\sigma_1^i}(N, v, L) + m_j^{\sigma_2^i}(N, v, L) \big) \\ &= \frac{1}{2n} \sum_{i \in N} \Big((v(\{i, i+1, \ldots, j\}) - v(\{i, i+1, \ldots, j-1\})) \\ &+ (v(\{i, i-1, \ldots, j\}) - v(\{i, i-1, \ldots, j+1\})) \Big) \\ &= \frac{1}{2n} \sum_{i \in N} \Big(v(S_i^j) - v(S_i^j \setminus \{j\}) + v(S_j^i) - v(S_j^i \setminus \{j\}) \Big). \end{aligned}$$

□

The lemma says that for a circle graph game the average tree solution for a player is the average of all his marginal contributions to any set containing all the players between him and any other player both clockwise and anticlockwise.

For a network $S \in C^L(N)$ a player $j \in S$ is called an end player of S if $\{j\} \cup (N \setminus S) \in C^L(N)$. The set of end players of S is denoted $E(S)$. Notice that $|E(S)| = 2$ if $1 < |S| < n$, and otherwise $|E(S)| = |S|$. Players that are in S but not in $E(S)$ are called middle players of S.

Theorem 3.4 *For any circle graph game $(N, v, L) \in \mathcal{G}^c$ the Shapley value as introduced in [2] coincides with the average tree solution, that is, $AT(N, v, L) = Sh(N, v, C^L(N))$.*

Proof. Take any $i \in N$. There are 2^{n-2} compatible permutations $\sigma \in \Pi^c(N)$ with $\sigma(1) = i$, yielding all marginal value $m_i^\sigma(N, v, L) = v(\{i\})$ for player i, and there are also 2^{n-2} compatible permutations $\sigma \in \Pi^c(N)$ with $\sigma(n) = i$, yielding all marginal value $m_i^\sigma(N, v, L) = v(N) - v(N \setminus \{i\})$ for player i. Now take $S \in C^L(N)$ with $i \notin S$ and $1 \leq |S| < n - 1$, and consider any compatible permutation $\sigma \in \Pi^c(N)$ such that $P_\sigma(i) = S$. This means that the first $|S|$ positions of σ are filled with the elements of S and the last positions of σ are filled with the elements not in $S \cup \{i\}$. Observe that either $i + 1$ or $i - 1$ is an end player of S since $S \cup \{i\} \in C^L(N)$ and $1 \leq |S| < n - 1$. For a permutation σ to be compatible, there are $2^{|S|-1}$ ways to fill the first $|S|$ positions of σ and there are $2^{n-|S|-2}$ ways to fill the last $n - |S| - 1$ positions of σ, which results in $2^{|S|-1} \cdot 2^{n-|S|-2} = 2^{n-3}$ compatible permutations σ with $P_\sigma(i) = S$. Any such permutation σ

yields marginal value $m_i^\sigma(N,v,L) = v(S\cup\{i\}) - v(S)$ for player i. The Shapley value of player i is the average of all these marginal contributions,

$$\begin{aligned}
Sh_i(N,v,C^L(N)) =& \frac{1}{2^{n-2}n} \sum_{\sigma\in\Pi^c(N)} m^\sigma(N,v,L) \\
=& \frac{1}{2^{n-2}n} \sum_{\sigma\in\Pi^c(N)} \Big(v(P_\sigma(i)\cup\{i\}) - v(P_\sigma(i))\Big) \\
=& \frac{1}{2^{n-2}n} \Big(\sum_{\substack{S\in C^L(N): S\cup\{i\}\in C^L(N)\\ 1\le |S|<n-1}} \Big(2^{n-3}(v(S\cup\{i\}) - v(S))\Big) \\
& + 2^{n-2}v(\{i\}) + 2^{n-2}(v(N) - v(N\backslash\{i\}))\Big) \\
=& \frac{1}{2n}\Big(\sum_{\substack{S\in C^L(N): i-1\in E(S)\\ 1\le |S|<n-1}} \Big(v(S\cup\{i\}) - v(S)\Big) + v(\{i\}) + v(N) - v(N\backslash\{i\})\Big) \\
& + \frac{1}{2n}\Big(\sum_{\substack{S\in C^L(N): i+1\in E(S)\\ 1\le |S|<n-1}} \Big(v(S\cup\{i\}) - v(S)\Big) + v(\{i\}) + v(N) - v(N\backslash\{i\})\Big) \\
=& \frac{1}{2n} \sum_{j\in N} \Big(v(S_j^i) - v(S_j^i\setminus\{i\}) + v(S_i^j) - v(S_i^j\setminus\{i\})\Big) \\
=& AT_i(N,v,L),
\end{aligned}$$

where the last equality holds due to Lemma 3.3. □

Although both solutions are different in terms of the number of marginal vectors to average, where the Shapley value takes the average of 2^{n-3} times more different marginal vectors than the average tree solution does, they coincide on the class of circle graph games. This is because each marginal contribution $v(S\cup\{i\}) - v(S)$ for some $i\in N$ and $S\in C^L(N)$ is counted only once for the average tree solution and 2^{n-3} times for the Shapley value. Since both solutions allocate the same payoff to each player on the class of circle graph games and the average tree solution counts every marginal contribution only once, we focus from now on on the average tree solution. The total payoff that the average tree solution assigns to a network can be expressed as follows.

Lemma 3.5 *For any circle graph game $(N,v,L)\in\mathcal{G}^c$, the payoff that the average tree solution allocates to a network S_a^b, $1\le a,b\le n$, is equal to*

$$AT_{S_a^b}(N,v,L) = \frac{|S_a^b|}{n} v(N) + \frac{1}{2n}\sum_{i\in N}\Big(v(S_a^i) - v(S_{b+1}^i) + v(S_i^b) - v(S_i^{a-1})\Big).$$

Proof. From Lemma 3.3 it follows that

$$\begin{aligned}
AT_{S_a^b}(N,v,L) =& \sum_{j=a}^{b}\Big(\frac{1}{2n}\sum_{i\in N}\Big(v(S_j^i)-v(S_j^i\setminus\{j\})+v(S_i^j)-v(S_i^j\setminus\{j\})\Big)\Big)\\
=& \sum_{j=a}^{b}\Big(\frac{1}{n}v(N)+\frac{1}{2n}\sum_{i\in N}\big(v(S_j^i)-v(S_{j+1}^i)+v(S_i^j)-v(S_i^{j-1})\big)\Big)\\
=& \frac{|S_a^b|}{n}v(N)+\frac{1}{2n}\sum_{i\in N}\sum_{j=a}^{b}\Big(v(S_j^i)-v(S_{j+1}^i)+v(S_i^j)-v(S_i^{j-1})\Big)\\
=& \frac{|S_a^b|}{n}v(N)+\frac{1}{2n}\sum_{i\in N}\Big(v(S_a^i)-v(S_{b+1}^i)+v(S_i^b)-v(S_i^{a-1})\Big).
\end{aligned}$$

□

Notice that the expression also holds when a is equal to b and the network is a single player.

Next we consider the average tree solution for unanimity circle graph games. Given a network $T\in C^L(N)$ as the minimum winning network, the triple $(N,u_T,L)\in\mathcal{G}^c$ is called the unanimity circle graph game for T, where $u_T(S)=1$ if $T\subset S$ and $u_T(S)=0$ otherwise. Herings *et al.* [8] interpret the average tree solution for cycle-free graph games by using the notion of representativeness of players. For a circle graph (N,L) and a network $S\in C^L(N)$, node $j\in S$ represents node $k\notin S$ for S if there is a path of nodes from j to k with all nodes except j outside S. The number of nodes represented by node $j\in N$ for network $S\in C^L(N)$ is expressed as $p_S^L(j)$, where $p_S^L(i)=0$ when $i\notin S$.

Theorem 3.6 *For any unanimity circle graph game $(N,u_T,L)\in\mathcal{G}^c$, $T\in C^L(N)$, it holds that*

$$AT_j(N,u_T,L)=\begin{cases}\frac{1}{n}\Big(1+\frac{p_T^L(j)}{|E(T)|}\Big) & j\in T\\ 0 & \textit{otherwise.}\end{cases}$$

Proof. Take $T\in C^L(N)$, then by definition $p_T^L(j)$ is equal to

$$p_T^L(j)=\begin{cases} n-|T| & j\in E(T)\\ 0 & \text{otherwise.}\end{cases}$$

Consider the case $T=\{i\}$ for some $i\in N$. Then $p_{\{i\}}^L(i)=n-1$ and $|E(\{i\})|=1$, so

$$AT_j(N,u_{\{i\}},L)=\begin{cases} 1=\frac{1}{n}\Big(1+\frac{p_{\{i\}}^L(j)}{|E(\{i\})|}\Big) & \text{if } j=i\\ 0 & \text{otherwise.}\end{cases}$$

Next, let $T=S_a^b\subsetneq N$ and without loss of generality assume $1\le a,b\le n$ and $a\neq b$. Notice that

$E(S_a^b) = \{a, b\}$ and $p^L_{S_a^b}(j) = n - |S_a^b|$ for $j \in E(S_a^b)$. For $j = a$ it follows from Lemma 3.5 that

$$\begin{aligned}
AT_j(N, u_{S_a^b}, L) &= \frac{1}{n}v(N) + \frac{1}{2n}\sum_{i\in N}\Big(u_{S_a^b}(S_a^i) - u_{S_a^b}(S_{a+1}^i) + u_{S_a^b}(S_i^a) - u_{S_a^b}(S_i^{a-1})\Big) \\
&= \frac{1}{n} + \frac{1}{2n}\Big(\sum_{i\in N} 1_{\{S_a^b \subset S_a^i\}} - \sum_{i\in N} 1_{\{S_a^b \subset S_{a+1}^i\}} + \sum_{i\in N} 1_{\{S_a^b \subset S_i^a\}} - \sum_{i\in N} 1_{\{S_a^b \subset S_i^{a-1}\}}\Big) \\
&= \frac{1}{n} + \frac{1}{2n}\Big(\sum_{i\in N} 1_{\{i \notin S_a^{b-1}\}} - \sum_{i\in N} 1_{\{i=a\}} + \sum_{i\in N} 1_{\{i=a+1\}} - \sum_{i\in N} 1_{\{i=a\}}\Big) \\
&= \frac{1}{n} + \frac{1}{2n}\big((n - |S_a^b| + 1) - 1 + 1 - 1\big) \\
&= \frac{1}{n}\big(1 + \frac{1}{2n}(n - |S_a^b|)\big) \\
&= \frac{1}{n}\Big(1 + \frac{p^L_{S_a^b}(j)}{|E(S_a^b)|}\Big).
\end{aligned}$$

The analogous calculation follows when $j = b$. For the case when $j \in S_{a+1}^{b-1}$ we have

$$\begin{aligned}
AT_j(N, u_{S_a^b}, L) &= \frac{1}{n}v(N) + \frac{1}{2n}\sum_{i\in N}\Big(u_{S_a^b}(S_j^i) - u_{S_a^b}(S_{j+1}^i) + u_{S_a^b}(S_i^j) - u_{S_a^b}(S_i^{j-1})\Big) \\
&= \frac{1}{n} + \frac{1}{2n}\Big(\sum_{i\in N} 1_{\{S_a^b \subset S_j^i\}} - \sum_{i\in N} 1_{\{S_a^b \subset S_{j+1}^i\}} + \sum_{i\in N} 1_{\{S_a^b \subset S_i^j\}} - \sum_{i\in N} 1_{\{S_a^b \subset S_i^{j-1}\}}\Big) \\
&= \frac{1}{n} + \frac{1}{2n}\Big(\sum_{i\in N} 1_{\{i=j-1\}} - \sum_{i\in N} 1_{\{i=j\}} + \sum_{i\in N} 1_{\{i=j+1\}} - \sum_{i\in N} 1_{\{i=j\}}\Big) \\
&= \frac{1}{n} + \frac{1}{2n}\big(1 - 1 + 1 - 1\big) \\
&= \frac{1}{n} \\
&= \frac{1}{n}\Big(1 + \frac{p^L_{S_a^b}(j)}{|E(S_a^b)|}\Big).
\end{aligned}$$

For the case when $j \notin S_a^b$ we have

$$\begin{aligned}
AT_j(N, u_{S_a^b}, L) &= \frac{1}{n}v(N) + \frac{1}{2n}\sum_{i\in N}\Big(u_{S_a^b}(S_j^i) - u_{S_a^b}(S_{j+1}^i) + u_{S_a^b}(S_i^j) - u_{S_a^b}(S_i^{j-1})\Big) \\
&= \frac{1}{n} + \frac{1}{2n}\Big(\sum_{i\in N} 1_{\{S_a^b \subset S_j^i\}} - \sum_{i\in N} 1_{\{S_a^b \subset S_{j+1}^i\}} + \sum_{i\in N} 1_{\{S_a^b \subset S_i^j\}} - \sum_{i\in N} 1_{\{S_a^b \subset S_i^{j-1}\}}\Big) \\
&= \frac{1}{n} + \frac{1}{2n}\Big(\sum_{i\in N} 1_{\{i \in S_b^{j-1}\}} - \sum_{i\in N} 1_{\{i \in S_b^j\}} + \sum_{i\in N} 1_{\{i \in S_{j+1}^a\}} - \sum_{i\in N} 1_{\{i \in S_j^a\}}\Big) \\
&= \frac{1}{n} + \frac{1}{2n}\big(|S_b^{j-1}| - |S_b^j| + |S_{j+1}^a| - |S_j^a|\big) \\
&= \frac{1}{n} + \frac{1}{2n}(-2) \\
&= 0.
\end{aligned}$$

Finally, consider the case $T = N$. Then $p^L_N(j) = 0$ for all $j \in N$, and so

$$AT_j(N, u_N, L) = \frac{1}{n} = \frac{1}{n}\Big(1 + \frac{p^L_N(j)}{|E(N)|}\Big),\ j \in N.$$

□

This result shows that in a unanimity circle graph game with minimum winning network T, $T \in C^L(N)$, each end player of T represents all the players not in T with equal share of the power $\frac{1}{n}$ of any such player. Thus the average tree solution of such a game is characterized as an allocation that assigns zero to the players outside T, the egalitarian payoff $\frac{1}{n}$ to the middle players of T, and an equal share of the remaining worth $(n-|T|+2)/n$ to each of the end players for representing the agents outside T. When T is the singleton $\{i\}$ player i gets all the payoff, if T is the grand coalition N each player gets the same payoff. Except for the unanimity game for the grand coalition and for a dictator, in which cases there are no middle players, the end players of the minimum winning network receive more payoff than the middle players due to the representativeness of these players for the players who are not in the network. This contrasts with the Myerson value, where for a unanimity circle graph game for coalition T all players in T receive the same payoff, $\frac{1}{|T|}$.

4 Axiomatic characterization

In this section, we axiomatize the average tree solution on the class of circle graph games. For the class of cycle-free graph games, Mishra and Talman [11] give a characterization of the average tree solution in contrast with the Shapley value of TU-games. Our objective is to see how different the characterization of the average tree solution would be if the underlying graph is circular. First we give the axioms we are going to use. The first three axioms are rather standard.

Definition 4.1 A solution $\pi : \mathcal{G}^c \to \mathbb{R}^n$ satisfies *efficiency* if for any circle graph game $(N, v, L) \in \mathcal{G}^c$,

$$\sum_{i \in N} \pi_i(N, v, L) = v(N).$$

A player $i \in N$ is called a dummy player in a circle graph game $(N, v, L) \in \mathcal{G}^c$ if this player never contributes, i.e., $v(S) - v(S \setminus \{i\}) = 0$ for all $S \in C^L(N)$ satisfying $S \setminus \{i\} \in C^L(N)$ and $i \in S$.

Definition 4.2 A solution $\pi : \mathcal{G}^c \to \mathbb{R}^n$ satisfies the *dummy property* if for any circle graph game $(N, v, L) \in \mathcal{G}^c$ and dummy player $i \in N$ in (N, v, L) it holds that $\pi_i(N, v, L) = 0$.

This property means that if a player never contributes when joining a network to which he is connected, his payoff must be zero.

For any two circle graph games (N, v, L) and (N, w, L) in $\mathcal{G}^c$ and $a, b \in \mathbb{R}$, the circle graph game $(N, av + bw, L)$ is well defined by $(av + bw)(S) = av(S) + bw(S)$ for all $S \in 2^N$.

Definition 4.3 A solution $\pi : \mathcal{G}^c \to \mathbb{R}^n$ satisfies *linearity* if for any two circle graph games (N, v, L) and (N, w, L) in $\mathcal{G}^c$ and any $a, b \in \mathbb{R}$ it holds that

$$\pi(N, av + bw, L) = a\pi(N, v, L) + b\pi(N, w, L).$$

Definition 4.4 A solution $\pi : \mathcal{G}^c \to \mathbb{R}^n$ satisfies *symmetry at end players in unanimity games* if for any unanimity circle graph game $(N, u_T, L) \in \mathcal{G}^c$ with $T \in C^L(N)$ it holds that $\pi_i(N, u_T, L) = \pi_j(N, u_T, L)$ for all $i, j \in E(T)$.

Symmetry at end players in unanimity games means that in unanimity circle graph games the end players of the minimum winning network must get the same payoff. In particular it implies that in the unanimity circle graph game for the grand coalition, every player gets the same payoff.

Definition 4.5 A solution $\pi : \mathcal{G}^c \to \mathbb{R}^n$ satisfies *independence in unanimity games* if for any unanimity circle graph games (N, u_T, L) and $(N, u_{T\cup\{j\}}, L)$ in $\mathcal{G}^c$ with $T, T \cup \{j\} \in C^L(N)$ and $j \notin T$, it holds for all $i \in T \setminus E(T)$ that $\pi_i(N, u_T, L) = \pi_i(N, u_{T\cup\{j\}}, L)$.

Independence in unanimity games means that if the minimum winning network of a unanimity circle graph game changes from T to T', $T \subset T'$, the payoff will not change for those who are middle players of T.

Theorem 4.6 *On the class of circle graph games, the average tree solution is the unique solution satisfying efficiency, the dummy property, linearity, symmetry at end players in unanimity games, and independence in unanimity games.*

Proof. First, we prove that the average tree solution satisfies these properties. Due to linearity, it suffices to show that the other properties only hold for unanimity circle graph games. For this class of games, it can be immediately seen that the average tree solution satisfies efficiency and the dummy property. Consider a unanimity circle graph game (N, u_T, L). From the proof of Theorem 3.6 it follows that the payoff of the average tree solution for the end players in T is the same because they have the same representation power. Also the representation power for each middle player i in T does not change when the minimum winning network changes from T to $T \cup \{j\}$, $j \notin T$, $T \cup \{j\} \in C^L(N)$, because in both cases it is equal to zero.
Second, we prove that for any unanimity circle graph game the average tree solution is the unique solution which satisfies efficiency, the dummy property, symmetry at end players in unanimity games, and independence in unanimity games. Let π be a solution which satisfies these axioms. The proof is done by induction on the size t of the network $T \in C^L(N)$ for the unanimity circle graph game (N, u_T, L). Consider the case where $t = n$ and the unanimity game is (N, u_N, L). By definition, every player is an end player. By efficiency and symmetry at end players in unanimity games all n players receive the same payoff with total payoff 1 and therefore

$$\pi_k(N, u_N, L) = \frac{1}{n} \quad \forall\, k \in N.$$

Since no player is representing some other player, $p^L_N(j) = 0$ for all $i \in N$ and we have

$$AT_k(N, u_N, L) = \frac{1}{n} \quad \forall\, k \in N.$$

Thus π is the average tree solution when $t = n$. Take $t < n$ and suppose π is the average tree solution for any unanimity circle graph game (N, u_S, L) with $|S| > t$, that is

$$\pi_k(N, u_S, L) = \begin{cases} \frac{1}{n}(1 + \frac{p_S^L(k)}{|E(S)|}) & \forall\, k \in S \\ 0 & \text{otherwise.} \end{cases}$$

We have to show that for any network $T \in C^L(N)$ with $|T| = t$ it holds that

$$\pi_k(N, u_T, L) = \begin{cases} \frac{1}{n}(1 + \frac{p_T^L(k)}{|E(T)|}) & \forall\, k \in T \\ 0 & \text{otherwise.} \end{cases}$$

Take a player $k \in N \setminus T$. For any network $S \in C^L(N)$ with $S \ni k$ and $S \setminus \{k\} \in C^L(N)$, we have $u_T(S) - u_T(S \setminus \{k\}) = 0$. The dummy axiom then yields

$$\pi_k(N, u_T, L) = 0 = AT_k(N, u_T, L).$$

Next, take a player $k \in T \setminus E(T)$. Since $T \neq N$ there exists $j \in N \setminus T$ satisfying $T \cup \{j\} \in C^L(N)$. Let $S = T \cup \{j\}$. By the axiom of independence in unanimity games and by induction,

$$\pi_k(N, u_T, L) = \pi_k(N, u_S, L) = AT_k(N, u_S, L) = \frac{1}{n}\Big(1 + \frac{p_S^L(k)}{|E(S)|}\Big).$$

In order to see that $\pi_k(N, u_T, L) = \frac{1}{n} = AT_k(N, u_T, L)$, we show that $p_S^L(k) = 0$. From $k \in T \setminus E(T)$ it follows that k is a middle player in T. If $S \neq N$ then k must also be a middle player in S, and if $S = N$ then k is an end player of S. In both cases we obtain $p_S^L(k) = 0$, and therefore

$$\pi_k(N, u_T, L) = \frac{1}{n}\Big(1 + \frac{p_S^L(k)}{|E(S)|}\Big) = \frac{1}{n} = \frac{1}{n}\Big(1 + \frac{p_T^L(k)}{|E(T)|}\Big) = AT_k(N, u_T, L).$$

Finally, take $k \in E(T)$. By efficiency and symmetry at end players in unanimity games, we obtain

$$\pi_k(N, u_T, L) = \frac{1}{|E(T)|}\Big(1 - \frac{|T| - |E(T)|}{n}\Big) = \frac{1}{n}\Big(1 + \frac{n - |T|}{|E(T)|}\Big).$$

From Theorem 3.6, we know that $p_T^L(k) = n - |T|$, and therefore

$$\pi_k(N, u_T, L) = \frac{1}{n}\Big(1 + \frac{n - |T|}{|E(T)|}\Big) = \frac{1}{n}\Big(1 + \frac{p_T^L(k)}{|E(T)|}\Big) = AT_k(N, u_T, L).$$

Thus $\pi(N, u_T, L) = AT(N, u_T, L)$ for any $T \in C^L(N)$ and therefore if a solution satisfies the four axioms on the class of unanimity circle graph games, it must be the average tree solution on this class. Then the proof is concluded since any circle graph game can be expressed as a linear combination of unanimity circle graph games and the average tree solution satisfies linearity. □

We see in Section 3 that the average tree solution coincides with the Shaply value introduced by Bilbao and Ordóñez [2] on the class of circle graph games. Thus on this class of games the average tree solution has the same characteristics as this value has. In their work, they characterize the value with the hierarchical strength axiom. As discussed in Section 1, Baron *et al.* [1] show that the average tree solution is the unique solution on the class of graph games that satisfies efficiency and $\mathcal{T}$-hierarchy. Both ideas come from the work of Faigle and Kern [5]. The former averages the hierarchical power of each agent in a feasible coalition among compatible orderings, while the latter does it among admissible orderings. Therefore both power measures average out and end up the same on the class of circle graph game.

5 Core properties

In this section, we study on the class of circle graph games the relationship between the average tree solution and the core. Especially, we provide necessary and sufficient conditions for each admissible marginal vector, for each compatible marginal vector, and for the average tree solution to be in the core, respectively. Shapley [15] shows that for a TU-game (N, v), the Shapley value is the barycenter of the core if the game is convex, i.e., $v(S) + v(T) \geq v(S \cup T) + v(S \cap T)$ for every $S, T \subset N$. If this inequality holds for every S, T with $S \cap T = \emptyset$, the game is called superadditive. For a circle graph game (N, v, L), we call it convex if its Myerson restricted game (N, v^L) is convex, and a superadditive circle graph game is defined similarly. We introduce a weaker form of convexity.

Definition 5.1 A circle graph game $(N, v, L) \in \mathcal{G}^c$ is *circular-convex* if

$$v(S) + v(T) \leq v(S \cup T) + v(S \cap T)$$

for any $S, T \in C^L(N)$ that satisfy at least one of the following conditions:

(1) $S \cup T = N$ and $S \cap T \in C^L(N)$;

(2) $S \cup T \in C^L(N)$ and $S \cap T = \emptyset$.

Condition (2) coincides with superadditivity. If $|N| \leq 3$ then circular-convexity coincides with convexity, because in that case a circle graph is the complete graph. For $|N| > 3$, circular-convexity is weaker than convexity, because it does not take the convex relationship into account between two non-disjoint networks S and T if $S \cup T \neq N$ or if $S \cap T$ consists of two components, i.e., when S and T overlap each other at both their ends. In other words, given a network S, networks to be considered as network T in Definition 5.1 are those which connect to S either clockwise or anticlockwise. For a network S the number of networks T satisfying the conditions (1) and (2) does not depend on the size of S as far as $1 \leq |S| < n$. The total number of different networks T with which circular-convexity has to be satisfied for S is equal to $2n - 1$, since, including N and $\emptyset$, there are $n+1$ networks which connect to S clockwise and also $n+1$ networks which connect to S anticlockwise, with $N, \emptyset$ and $N \setminus S$ being counted twice. This number is the same for convexity only if $|S| = 1$. If $|S| > 1$, this number is larger for convexity than $2n - 1$ and also depends on the size of S. Circular-convexity is therefore stronger than superadditivity, but weaker than convexity.

We show that circular-convexity of a circle graph game is a necessary and sufficient condition for all admissible marginal vectors to be in the core.

Theorem 5.2 *For a circle graph game $(N, v, L) \in \mathcal{G}^c$, every admissible marginal vector $m^\sigma(N, v, L)$, $\sigma \in \Pi^a(N)$, is in the core if and only if the game is circular-convex.*

Proof. Suppose the circle graph game (N, v, L) is circular-convex. Take any network $S \in C^L(N)$. We show that for every $\sigma \in \Pi^a(N)$, $m^\sigma(S) \geq v(S)$, where $m^\sigma(S) = \sum_{j \in S} m_j^\sigma(N, v, L)$. Since $S \in C^L(N)$, we have $S = S_a^b$ for some $1 \leq a, b \leq n$. Without loss of generality, we assume $a \leq b$ and $\sigma = \sigma_1^i$ for some $i \in N$. Recall that σ_1^i is the admissible permutation with $\sigma(1) = i$ and $\sigma(n) = i - 1$. First suppose $i \notin S$, that is, either $1 \leq i < a$ or $b < i \leq n$. Then it holds that

$$\begin{aligned}
m^{\sigma_1^i}(S_a^b) &= v(S_i^b) - v(S_i^{a-1}) \\
&= v(S_i^{a-1} \cup S_a^b) - v(S_i^{a-1}) \\
&\geq v(S_i^{a-1}) + v(S_a^b) - v(S_i^{a-1}) \\
&= v(S_a^b),
\end{aligned}$$

where the inequality follows from condition (2) of Definition 5.1. Next, suppose $i \in S$. If $i = a$, then

$$m^{\sigma_1^i}(S_a^b) = v(S_a^b) - v(\emptyset) = v(S_a^b).$$

If $a < i \leq b$, then

$$\begin{aligned}
m^{\sigma_1^i}(S_a^b) &= m^{\sigma_1^i}(S_i^b) + m^{\sigma_1^i}(S_a^{i-1}) \\
&= v(S_i^b) - v(\emptyset) + v(S_i^{i-1}) - v(S_i^{a-1}) \\
&= v(S_i^b) + v(N) - v(S_i^{a-1}) \\
&= v(S_i^{a-1} \cap S_a^b) + v(S_i^{a-1} \cup S_a^b) - v(S_i^{a-1}) \\
&\geq v(S_a^b),
\end{aligned}$$

where the inequality follows from condition (1) of Definition 5.1. Therefore circular-convexity is a sufficient condition.

Suppose that $m^\sigma(N, v, L) \in C(N, v, L)$ for every $\sigma \in \Pi^a(N)$, but that the circle graph game (N, v, L) is not satisfying circular-convexity. Then there are two distinct networks S and T which satisfy at least one of the conditions of Definition 5.1 while $v(S) + v(T) > v(S \cup T) + v(S \cap T)$. First, consider the case when Condition (2) of Definition 5.1 holds, i.e., $S \cup T \in C^L(N)$ and $S \cap T = \emptyset$. Without loss of generality, let $S = S_a^b$ and $T = S_{b+1}^c$ with $1 \leq a \leq b < c \leq n$. Then it holds for the marginal vector $m^{\sigma_2^c}$ that

$$\begin{aligned}
m^{\sigma_2^c}(S_a^b) &= v(S_a^c) - v(S_{b+1}^c) \\
&= v(S_a^b \cup S_{b+1}^c) - v(S_{b+1}^c) \\
&= v(S \cup T) - v(T) \\
&< v(S) = v(S_a^b).
\end{aligned}$$

This contradicts that $m^{\sigma_2^c}(N, v, L) \in C(N, v, L)$. Next, consider the case when Condition (1) of Definition 5.1 holds, i.e., $S \cup T = N$ and $S \cap T \in C^L(N)$. Without loss of generality, let $S = S_a^b$

and $T = S_c^{a-1}$ with $1 \le a < c \le b \le n$. Observe that $S \cap T = S_c^b$. Then for the marginal vector $m^{\sigma_1^c}$ it holds that

$$\begin{aligned}
m^{\sigma_1^c}(S_a^b) &= m^{\sigma_1^c}(S_c^b) + m^{\sigma_1^c}(S_a^{c-1}) \\
&= v(S_c^b) - v(\emptyset) + v(S_c^{c-1}) - v(S_c^{a-1}) \\
&= v(S_c^b) + v(N) - v(S_c^{a-1}) \\
&= v(S_c^{a-1} \cap S_a^b) + v(S_c^{a-1} \cup S_a^b) - v(S_c^{a-1}) \\
&= v(S \cap T) + v(S \cup T) - v(T) \\
&< v(S) = v(S_a^b).
\end{aligned}$$

This contradicts that $m^{\sigma_1^c}(N, v, L) \in C(N, v, L)$. It is shown that whenever there is a violation for circular-convexity, there is an admissible marginal vector outside the core. This concludes that circular-convexity is also a necessary condition. $\square$

From the theorem it immediately follows that the convex hull of all admissible marginal vectors is a subset of the core if and only if the game is circular-convex.

Corollary 5.3 *For any circle graph game $(N, v, L) \in \mathcal{G}^c$ it holds that Conv $\{m^\sigma(N, v, L) | \sigma \in \Pi^a(N)\}$ is a subset of the core $C(N, v, L)$ if and only if the game is circular-convex.*

Since the average tree solution is the average of all admissible marginal vectors, we have the following corollary.

Corollary 5.4 *For any circular-convex circle graph game $(N, v, L) \in \mathcal{G}^c$ it holds that the core is nonempty and that $AT(N, v, L) \in C(N, v, L)$.*

We need a stronger condition than circular-convexity for the stability of all compatible marginal vectors, since the number of marginal vectors to be considered is 2^{n-3} times more and all marginal vectors are different from each other.

Definition 5.5 A circle graph game $(N, v, L) \in \mathcal{G}^c$ is *strongly circular-convex* if

$$v(S) + v(T) \le v(S \cup T) + v(S \cap T)$$

for any $S, T \in C^L(N)$ such that $S \cup T \in C^L(N)$ and $S \cap T \in C^L(N)$.

Note that for $|N| \le 3$, strong circlular-convexity is equivalent to convexity. If $|N| > 3$, the condition is waker than convexity but stronger than circular-convexity, since it also requires a convex relationship between each non-disjoint pair of networks S and T with $S \cup T \ne N$.

Theorem 5.6 *For a circle graph game $(N, v, L) \in \mathcal{G}^c$, every compatible marginal vector $m^\sigma(N, v, L)$, $\sigma \in \Pi^c(N)$, is in the core if and only if the game is strongly circular-convex.*

Proof. Suppose the circle graph game (N, v, L) is strongly circular-convex. Take any network $S \in C^L(N)$. We show that for every $\sigma \in \Pi^c(N)$ it holds that $m^\sigma(S) - v(S) \ge 0$, where $m^\sigma(S) =$

$\sum_{j\in S} m_j^\sigma(N,v,L)$. Without loss of generality, let us order the players in S on σ as $i_1,\ldots,i_{|S|}$ so that $a<b$ implies $P_\sigma(i_a)\subset P_\sigma(i_b)$. Note that this ordering is uniquely determined given σ and S. Then, from equation (3.1), we have

$$\begin{aligned}
m^\sigma(S)-v(S) =&\sum_{k=1}^{|S|}\Big(v\big(P_\sigma(i_k)\cup\{i_k\}\big)-v\big(P_\sigma(i_k)\big)\Big)-v(S)\\
=&\sum_{k=1}^{|S|-1}\Big(v\big(P_\sigma(i_k)\cup\{i_k\}\big)-v\big(P_\sigma(i_k)\big)\Big)+v\big(P_\sigma(i_{|S|})\cup\{i_{|S|}\}\big)-v\big(P_\sigma(i_{|S|})\big)-v(S)\\
=&\sum_{k=1}^{|S|-1}\Big(v\big(P_\sigma(i_k)\cup\{i_k\}\big)-v\big(P_\sigma(i_k)\big)\Big)+v\big(P_\sigma(i_{|S|})\cup S\big)-v\big(P_\sigma(i_{|S|})\big)-v(S)\\
\geq&\sum_{k=1}^{|S|-1}\Big(v\big(P_\sigma(i_k)\cup\{i_k\}\big)-v\big(P_\sigma(i_k)\big)\Big)-v\big(P_\sigma(i_{|S|})\cap S\big)\\
=&\sum_{k=1}^{|S|-1}\Big(v\big(P_\sigma(i_k)\cup\{i_k\}\big)-v\big(P_\sigma(i_k)\big)\Big)-v(S\setminus\{i_{|S|}\}),
\end{aligned}$$

where the inequality follows from the strong circular-convexity condition for networks S and $P_\sigma(i_{|S|})$. Notice that $P_\sigma(i_{|S|})\cup S=P_\sigma(i_{|S|})\cup\{i_{|S|}\}$ and $P_\sigma(i_{|S|})\cap S=S\setminus\{i_{|S|}\}$. Repeating the procedure gives

$$\begin{aligned}
m^\sigma(S)-v(S)\geq&\sum_{k=1}^{|S|-1}\Big(v\big(P_\sigma(i_k)\cup\{i_k\}\big)-v\big(P_\sigma(i_k)\big)\Big)-v(S\setminus\{i_{|S|}\})\\
\geq&\sum_{k=1}^{|S|-2}\Big(v\big(P_\sigma(i_k)\cup\{i_k\}\big)-v\big(P_\sigma(i_k)\big)\Big)-v(S\setminus\{i_{|S|-1},i_{|S|}\})\\
\cdots&\\
\geq&\Big(v\big(P_\sigma(i_1)\cup\{i_1\}\big)-v\big(P_\sigma(i_1)\big)\Big)-v(S\setminus\{i_2,\ldots,i_{|S|}\})\\
=&v\big(P_\sigma(i_1)\cup\{i_1\}\big)-v\big(P_\sigma(i_1)\big)-v(\{i_1\})\\
\geq&0.
\end{aligned}$$

Therefore strong circular-convexity is a sufficient condition.

Suppose that $m^\sigma(N,v,L)\in C(N,v,L)$ for every $\sigma\in\Pi^c(N)$, but that the circle graph game (N,v,L) is not satisfying strong circular-convexity. Then there are two distinct networks S and T which satisfy the conditions of Definition 5.5 while $v(S)+v(T)>v(S\cup T)+v(S\cap T)$. From the proof of Theorem 5.2, if $S\cup T\in C^L(N)$ and $S\cap T=\emptyset$ or if $S\cup T=N$ and $S\cap T\in C^L(N)$, there is an admissible, therefore compatible, permutation σ satisfying $m^\sigma(N,v,L)\notin C(N,v,L)$, which is a contradiction. Thus it suffices to find a compatible permutation σ with $m^\sigma(N,v,L)\notin C(N,v,L)$ whenever $S\cap T\in C^L(N)$ and $S\cup T\neq N$. Take any compatible permutation σ with the first $|S\cap T|$ positions occupied by the elements of $S\cap T$, the next $|T\setminus S|$ positions occupied by the elements of $T\setminus S$, and the next $|S\setminus T|$ positions occupied by the elements of $S\setminus T$. Such a compatible permutation exists, because $S \cap T$, $T \setminus S$, $S \setminus T$ and $N \setminus (S \cup T)$ are nonempty networks while $S \cap T$ and $T \setminus S$, also $(S \cap T) \cup (T \setminus S) = T$ and $S \setminus T$, and also $(S \cap T) \cup (T \setminus S) \cup (S \setminus T) = S \cup T$ and $N \setminus (S \cup T)$ are disjoint but connected. Observe that σ cannot be admissible. It holds that $m^\sigma(S \cap T) = v(S \cap T)$ and $m^\sigma(S \setminus T) = m^\sigma(S \cup T) - m^\sigma(T) = v(S \cup T) - v(T)$. Then it follows that $m^\sigma(S) = m^\sigma(S \cap T) + m^\sigma(S \setminus T) = v(S \cap T) + v(S \cup T) - v(T) < v(S)$ and therefore $m^\sigma(N, v, L) \notin C(N, v, L)$. $\square$

Corollary 5.7 *For any circle graph game $(N, v, L) \in \mathcal{G}^c$ it holds that $Conv\,\{m^\sigma(N, v, L) | \sigma \in \Pi^c(N)\}$ is a subset of the core $C(N, v, L)$ if and only if the graph game is strongly circular-convex.*

The next examples illustrate that these convexity conditions are indeed weaker than convexity.

Example 5.8 Consider the 4-person circle graph game (N, v, L) with characteristic function

$$v(S) = \begin{cases} 2 & \text{if} \quad S = N, \{1,3,4\}, \\ 1 & \text{if} \quad S = \{1,4\}, \{3,4\}, \{1,2,3\}, \{1,2,4\}, \{2,3,4\}, \\ 0 & \text{otherwise.} \end{cases}$$

This game is strongly circular-convex but not convex (take $S = \{1,2,3\}$ and $T = \{1,3,4\}$). From Theorem 5.6, it follows that $m^\sigma(N, v, L) \in C(N, v, L)$ for all $\sigma \in \Pi^c(N)$. Observe that player 2 is a dummy player and therefore a stable allocation assigns to this player zero payoff. Since player 2 is not a dummy player in the Myerson restricted game (N, v^L) $\big(v^L(\{1,2,3\}) - v^L(\{1,3\}) = 1 > 0\big)$, the Myerson value allocates some positive value to this player, yielding $\mu(N, v, L) = (\frac{7}{12}, \frac{1}{12}, \frac{7}{12}, \frac{9}{12}) \notin C(N, v, L) = C(N, v^L)$. The average tree solution equals $AT(N, v, L) = (\frac{5}{8}, 0, \frac{5}{8}, \frac{6}{8}) \in C(N, v, L)$.

Example 5.9 Consider the 4-person circle graph game (N, v, L) with characteristic function

$$v(S) = \begin{cases} 2 & \text{if} \quad S = N, \\ 2 - \epsilon & \text{if} \quad S = \{1,3,4\}, \\ 1 & \text{if} \quad S = \{1,4\}, \{3,4\}, \{1,2,3\}, \{1,2,4\}, \{2,3,4\}, \\ 0 & \text{otherwise,} \end{cases}$$

for some $0 \leq \epsilon \leq 1$. For $\epsilon = 0$ this is Example 5.8. For $0 < \epsilon \leq 1$, the game is circular-convex but not strongly-circular-convex (take $S = \{1,4\}$ and $T = \{3,4\}$). From Theorem 5.2 it follows that $m^\sigma(N, v, L) \in C(N, v, L)$ for all $\sigma \in \Pi^a(N)$. For the permutation $\sigma = (4,3,1,2) \in \Pi^c(N) \setminus \Pi^a(N)$, however, it holds that $m^\sigma(N, v, L) = (1-\epsilon, \epsilon, 1, 0) \notin C(N, v, L)$, since $m_1^\sigma(N, v, L) + m_4^\sigma(N, v, L) < v(\{1,4\})$. The average tree solution equals $AT(N, v, L) = (\frac{5-\epsilon}{8}, \frac{2\epsilon}{8}, \frac{5-\epsilon}{8}, \frac{6}{8}) \in C(N, v, L)$ and the Myerson value equals $\mu(N, v, L) = (\frac{7-\epsilon}{12}, \frac{1+3\epsilon}{12}, \frac{7-\epsilon}{12}, \frac{9-\epsilon}{12})$. Note that $\mu(N, v, L) \notin C(N, v, L)$ for $0 \leq \epsilon < \frac{1}{9}$.

Next, we proceed to find a necessary and sufficient condition for the average tree solution on the class of circle graph games to be stable. The value is the average of admissible marginal vectors that are stable under circular-convexity, and the desired condition can be seen as the one under which circular-convexity is, on average, satisfied. For a nonempty network $S \in C^L(N), S \neq N$, of a circle graph (N, L), let $C^1(S)$ be the collection of networks that are connected to S clockwise and let $C^2(S)$ be the collection of networks that are connected to S anticlockwise, i.e., with $S = S_a^b$ for some $1 \leq a, b \leq n$, $C^1(S) = \{S_{b+1}^j | j \in N\}$ and $C^2(S) = \{S_j^{a-1} | j \in N\}$. Note that $|C^1(S)| = |C^2(S)| = n$.

Definition 5.10 A circle graph game $(N, v, L) \in \mathcal{G}^c$ is *average-circular-convex* if for any $S \in C^L(N)$, $S \neq N$, it holds that

$$\sum_{k=1}^{2} \sum_{T \in C^k(S)} (v(S) + v(T)) \leq \sum_{k=1}^{2} \sum_{T \in C^k(S)} (v(S \cup T) + v(S \cap T)).$$

Theorem 5.11 *For any circle graph game $(N, v, L) \in \mathcal{G}^c$ average-circular-convexity is a necessary and sufficient condition for the average tree solution to be stable.*

Proof. First, we rewrite the condition in Definition 5.10 as

$$\sum_{k=1}^{2} \sum_{T \in C^k(S)} \Big(v(S \cup T) + v(S \cap T) - v(S) - v(T) \Big) \geq 0.$$

Due to efficiency, $AT_N(N, v, L) = v(N)$. Now, take any network S with $|S| < n$ and without loss of generality let $S = S_a^b$ for some $1 \leq a \leq b \leq n$. From Lemma 3.5 we have

$$\begin{aligned}
AT_{S_a^b}(N, v, L) =& \frac{|S_a^b|}{n} v(N) + \frac{1}{2n} \sum_{i \in N} \Big(v(S_a^i) - v(S_{b+1}^i) + v(S_i^b) - v(S_i^{a-1}) \Big) \\
=& \frac{1}{2n} \Big[\sum_{i \notin S} \Big(v(S_a^i) - v(S_{b+1}^i) + v(S_i^b) - v(S_i^{a-1}) \Big) \\
&+ \sum_{i \in S} \Big(v(N) + v(S_a^i) - v(S_{b+1}^i) + v(N) + v(S_i^b) - v(S_i^{a-1}) \Big) \Big] \\
=& \frac{1}{2n} \Big[\sum_{i \notin S} \Big(v(S_a^b \cup S_{b+1}^i) - v(S_{b+1}^i) + v(S_a^b \cup S_i^{a-1}) - v(S_i^{a-1}) \Big) \\
&+ \sum_{i \in S} \Big(v(S_a^b \cup S_{b+1}^i) + v(S_a^i) - v(S_{b+1}^i) + v(S_a^b \cup S_i^{a-1}) + v(S_i^b) - v(S_i^{a-1}) \Big) \Big] \\
=& \frac{1}{2n} \Big[\sum_{i \in N} \Big(v(S_a^b \cup S_{b+1}^i) + v(S_a^b \cap S_{b+1}^i) - v(S_{b+1}^i) \Big) \\
&+ \sum_{i \in N} \Big(v(S_a^b \cup S_i^{a-1}) + v(S_a^b \cap S_i^{a-1}) - v(S_i^{a-1}) \Big) \Big] \\
=& \frac{1}{2n} \Big[\sum_{T \in C^1(S)} \Big(v(S_a^b \cup T) + v(S_a^b \cap T) - v(T) \Big) \\
&+ \sum_{T \in C^2(S)} \Big(v(S_a^b \cup T) + v(S_a^b \cap T) - v(T) \Big) \Big].
\end{aligned}$$

Therefore it follows that

$$AT_S(N, v, L) = \frac{1}{2n} \sum_{k=1}^{2} \sum_{T \in C^k(S)} \Big(v(S \cup T) + v(S \cap T) - v(T) \Big).$$

Then,

$$\begin{aligned}AT_S(N,v,L)-v(S)=&\frac{1}{2n}\sum_{k=1}^{2}\sum_{T\in C^k(S)}\Big(v(S\cup T)+v(S\cap T)-v(T)\Big)-v(S)\\=&\frac{1}{2n}\sum_{k=1}^{2}\sum_{T\in C^k(S)}\Big(v(S\cup T)+v(S\cap T)-v(S)-v(T)\Big),\end{aligned}$$

which is nonnegative if and only if average-circular-convexity is satisfied. □

Notice that average-circular-convexity is weaker than circular-convexity. It is also observed that the average tree solution of a network S can be expressed as the average of $2n$ convex surpluses $(v(S\cup T)+v(S\cap T)-v(T))$ for the networks that are connected to S clockwise or anticlockwise. The convex surplus for $T=N$, $v(S)$, is counted twice, representing the contribution of S with the two admissible permutations in which the first $|S|$ places are filled with players from S ordered either clockwise or anticlockwise. Also the convex surplus for $T=N\setminus S$, $v(N)-v(N\setminus S)$, is counted twice, since S can connect to its complement by using either of two links.

Since average-circular-convexity implies that the average tree solution is stable, we obtain the following corollary.

Corollary 5.12 *If a circle graph game $(N,v,L)\in\mathcal{G}^c$ is average-circular-convex, then the core $C(N,v,L)$ is nonempty.*

The next example shows that average-circular-convexity does not imply superadditivity.

Example 5.13 Consider the 4-person circle graph game (N,v,L) with characteristic function

$$v(S)=\begin{cases}2 & \text{if} \quad S=N,\{1,3,4\},\\ 1+\epsilon & \text{if} \quad S=\{3,4\},\{1,4\},\{1,2,3\},\\ 1 & \text{if} \quad S=\{1,2,4\},\{2,3,4\},\\ \epsilon & \text{if} \quad S=\{1,2\},\{2,3\},\\ 0 & \text{otherwise,}\end{cases}$$

for some $0\le\epsilon\le\frac{1}{3}$. For $\epsilon=0$ this is Example 5.8. For any $\epsilon>0$, this game is not superadditive (take $S=\{2\}$ and $T=\{3,4\}$) and in particular, it can be checked that every marginal vector is outside the core, i.e., $m^\sigma(N,v,L)\notin C(N,v,L)$ for all $\sigma\in\Pi(N)$. However, for $0\le\epsilon\le\frac{1}{3}$, this game is average-circular-convex and the average tree solution equals $AT(N,v,L)=(\frac{5+\epsilon}{8},0,\frac{5+\epsilon}{8},\frac{6-2\epsilon}{8})\in C(N,v,L)$, while the Myerson value is never in the core, $\mu(N,v,L)=(\frac{7+\epsilon}{12},\frac{1+\epsilon}{12},\frac{7+\epsilon}{12},\frac{9-3\epsilon}{12})\notin C(N,v,L)$.

6 Conclusion

In this paper we study two solution concepts on the class of circle graph games in which the nodes are located on a circle and communication only takes place between neighboring nodes.

The first one is the average of marginal vectors induced from permutations in which each player is connected to the player preceding him in the permutation, which turns out to be the average tree solution. The other one is the average of marginal vectors induced from permutations in which each player is connected to one of the players preceding him in the permutation, which is identical to the Shapley value introduced by Bilbao and Ordóñez in [2]. Although both solutions in general differ from each other, it is known that in the class of full communication games, they coincide with the Shapley value. We show that also on the class of circle graph games the two solution concepts coincide although the second solution is the average of many more different marginal vectors. Both solutions lead to the same payoff distribution that allocates to each player the average of his marginal contributions to all the networks that he is connected to. With the expression of this solution on the class of unanimity circle graph games, we also give an axiomatic characterization of the solution on the class of circle graph games and show that the payoff of a player is equal to the sum of his dividends over all networks to which he belongs, where the solution distributes to him a payoff expressed by his representation power. Concerning stability, we introduce three notions of convexity that are necessary and sufficient to guarantee that all marginal vectors corresponding to the each of the two solutions and the solution itself are elements of the core, respectively.

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